1.2 Sobolev space

1.2.1 Integral, metric and partition of unity

The main purpose of this note is to study the differential operator. In Definition 1.1.14, a differential operator is a linear map

$$P : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, F). \tag{1.2.1}$$

Once we see the linear map, the want to use the linear algebra, the matrix theory to study it. Unfortunately, in general, $\mathscr{C}^\infty(M, E)$ and $\mathscr{C}^\infty(M, F)$ are infinite dimensional vector space. The tool of studying the infinite dimensional vector spaces is the functional analysis. So naturally we plan to use the functional analysis to study the differential operator. In order to use the functional analysis, we firstly need to define an inner product on $\mathscr{C}^\infty(M, E)$ and extend it to the Hilbert space. After all, the theory of functional analysis we know for the undergraduates are based on the Hilbert space.

How to define a Hermitian product on $\mathscr{C}^\infty(M, E)$?

We first study it for $M = \mathbb{R}^n$, $E = \mathbb{C}$. For $f, g \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{C})$, the classical Hermitian product is defined by

$$\langle f, g\rangle := \int_{\mathbb{R}^n} f(x) \cdot \overline{g(x)} dv_x. \tag{1.2.2}$$

Note that the right hand side of (1.2.2) might be infinity. We denote by

$$\|f\|^2_{L^2} := \langle f, f\rangle. \tag{1.2.3}$$

Let $L^2(\mathbb{R}^n)$ be the completion of the set $\{f \in \mathscr{C}^0(\mathbb{R}^n, \mathbb{C}) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\|\cdot\|_{L^2}$. It is a Hilbert space.

Let the support of f, $\mathrm{supp}(f)$, be the closure of

$$\{x \in \mathbb{R}^n : f(x) \neq 0\}. \tag{1.2.4}$$

We denote by $\mathscr{C}^\infty_0(M, \mathbb{C})$ be the set of smooth functions with compact support. It is easy to see that $\mathscr{C}^\infty_0(M, \mathbb{C}) \subset L^2(\mathbb{R}^n)$. Moreover, we all know that the completion of $\mathscr{C}^\infty_0(M, \mathbb{C})$ with respect to the norm $\|\cdot\|_{L^2}$ is $L^2(\mathbb{R}^n)$. Similarly, we denote by $\mathscr{C}^\infty_0(M, E)$ be the set of smooth sections with compact support.

The next step is to define the Hermitian product on $\mathscr{C}^\infty(M, \mathbb{C})$ and $L^2(M)$. Naturally, we want to follow the definition in (1.2.2). The problem is

How to define the integration on a manifold?

As in (1.2.4), for $f \in \mathscr{C}^\infty(M, \mathbb{C})$, we define

$$\operatorname{supp}(f) := \overline{\{x \in M : f(x) \neq 0\}}. \tag{1.2.5}$$

For a chart U_j of M, if $\operatorname{supp}(f) \subset U_j$, naturally, we can define

$$\int_M f dv := \int_{\mathbb{R}^n} f \circ \phi_j^{-1}(x) dv_x^{(j)}. \tag{1.2.6}$$

As usual, we need to check it for another chart. If $\operatorname{supp}(f) \subset U_i \cap U_j$, in chart U_i, by coordinate transformation formula,

$$\begin{aligned}\int_{\mathbb{R}^n} f \circ \phi_i^{-1}(x) dv_x^{(i)} &= \int_{\mathbb{R}^n} f \circ \phi_i^{-1}(\phi_{ij}(x)) |\det(\Phi_{ij})| dv_x^{(j)} \\ &= \int_{\mathbb{R}^n} f \circ \phi_j^{-1}(x) |\det(\Phi_{ij})| dv_x^{(j)}. \end{aligned} \tag{1.2.7}$$

The annoying term $|\det(\Phi_{ij})|$ prevents us defining the integral over a manifold in a natural way. We overcome it with the same method as the vector field: glue them by a transformation relation together to get a vector bundle, then take the section to do things.

Definition 1.2.1. We define the density bundle $|\Lambda|$ over M by the transition function $\Psi_{ij} = |\det(\Phi_{ij})|^{-1}$. It is a 1-dimensional real vector bundle.

Then the integral over M could be defined as a linear form

$$\int_M : \mathscr{C}_0^\infty(M, |\Lambda|) \to \mathbb{R}. \tag{1.2.8}$$

This idea is reasonable. But it is a little abstract. So we'll not go this way.

From (1.1.29), if all $\det(\Phi_{ij}) > 0$, we see that $|\Lambda| = \Lambda^n T^* M$. We are familiar with $\Lambda^n T^* M$. So we are shamed to assume in this note that all $\det(\Phi_{ij}) > 0$. Now we give it a name.

Definition 1.2.2. We say M is oriented if there exists an atlas such that for any $U_i \cap U_j \neq \emptyset$, $\det(\Phi_{ij}) > 0$.

In this note, we always assume that M is oriented.

From this point of view, we see that the integral is more natural defined on n-forms than the smooth functions.

For $\alpha \in \mathscr{C}^\infty(M, \Lambda^n T^* M)$ such that $\operatorname{supp}(\alpha) \subset U_i$, from the argument above, if on U_i, $\alpha = f \cdot dx_1^{(i)} \wedge \cdots \wedge dx_n^{(i)}$, we see that the definition

$$\int_M \alpha := \int_{\mathbb{R}^n} f \cdot dx_1^{(i)} \cdots dx_n^{(i)} \tag{1.2.9}$$

is meaningful.

As in (1.1.24), in multivariable calculus, the term $dx_1 \cdots dx_n$ could be explained as then n-form $dx_1 \wedge \cdots \wedge dx_n$.

Now we want to integrate the n-form on the whole manifold. We introduce the trick of partition of unity.

Partition of unity

In the definition of manifold, we assume that M is second countable. From the knowledge of the topology, it implies that M is paracompact, that is, each open covering of M admits a locally finite refinement. Thus in the followings, we always assume that our covering of the atlas is locally finite, i.e., each point only lives in finite charts.

Theorem 1.2.3 (Partition of unity)**.** *There exists a family of smooth functions* $\{\varphi_i\}$ *such that* $\operatorname{supp}(\varphi_i) \subset U_i$ *and*

$$\sum_i \varphi_i(x) = 1. \tag{1.2.10}$$

Remark that since we assume that $\{U_i\}$ is locally finite, for each $x \in M$, the sum in (1.2.10) is a finite sum.

Proof. We could choose an open covering $\{V_i\}$ such that $\overline{V_i} \subset U_i$. Then we could construct functions $g_i \in \mathscr{C}^\infty(M, \mathbb{R})$ such that $V_i \subset \operatorname{supp}(g_i) \subset U_i$. Thus $g(x) := \sum_i g_i(x) > 0$ for any $x \in M$. Then we could take $\varphi_i := g_i/g$. □

For a n-form α, we have $\operatorname{supp}(\varphi_i \cdot \alpha) \in U_i$. Thus from (1.2.9), we could define

$$\int_M \alpha = \int_M (\sum_i \varphi_i(x))\alpha = \sum_i \int_M \varphi_i \cdot \alpha. \tag{1.2.11}$$

Note that our functions of partition of unity are not unique. We need to check our definition in (1.2.11) does not depend on the choice of the partition of unity. It is left to the reader.

Until now, we obtain the definition of the integration of a n-form. The definition is naturally extended to the integration of any differential form by taking $\int_M \beta = 0$ for any $\beta \in \mathscr{C}^\infty(M, \Lambda^k T^*M)$ for $k < n$.

Since we want to extend (1.2.2) to the manifold, we also need to define the integration of a function.

From the construction of $\Lambda^n T^*M$, there exists nowhere vanishing n-form (not unique) on M. Such a nowhere vanishing n-form is called a volume form, usually denoted by dv_x.

Remark that the existence of the nowhere vanishing n-form implies that $\Lambda^n T^*M$ is a 1-dimensional trivial vector bundle (Since M is oriented).

After taking a volume form, we could define the integration of a function f by taking the integration of $f\,dv_x \in \mathscr{C}^\infty(M, \Lambda^n T^*M)$.

Now for $f, g \in \mathscr{C}^\infty(M, \mathbb{C})$, the classical Hermitian product is defined by

$$\langle f, g\rangle := \int_M f(x)\cdot \overline{g(x)} dv_x. \tag{1.2.12}$$

We denote the norm by

$$\|f\|_{L^2}^2 := \langle f, f\rangle. \tag{1.2.13}$$

Let $L^2(M)$ be the completion of the set $\{f \in \mathscr{C}^\infty(M, \mathbb{C}) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\|\cdot\|_{L^2}$. It is also a Hilbert space.

The next question is how to do these things for sections of a vector bundle?

The key point is how to do $f(x)\cdot\overline{g(x)}$ for sections.

For vector bundle, the fiber is a vector space. Let $\pi : E \to M$ be the projection. For each $x \in M$, $E_x := \pi^{-1}(x)$ is a complex vector space and $f(x), g(x) \in E_x$ are vectors. From the knowledge of linear algebra, if there is a Hermitian inner product $\langle\cdot,\cdot\rangle_x$ on E_x, we could replace $f(x)\cdot\overline{g(x)}$ by $\langle f(x), g(x)\rangle_x$. We also need the inner product $\langle\cdot,\cdot\rangle_x$ depends smoothly on x. Usually, we also denote by

$$h_x^E(f(x), g(x)) := \langle f(x), g(x)\rangle_x. \tag{1.2.14}$$

Note that $h_x^E(\cdot,\cdot)$ is linear on the first variable and conjugate linear on the second one. Such map is called the sesquilinear map.

Definition 1.2.4. The Hermitian metric is a smooth family $\{h_x^E\}_{x\in M}$ of sesquilinear maps $h_x^E : E_x \times E_x \to \mathbb{C}$ such that $h_x^E(\xi,\xi) > 0$ for any $\xi \in E_x\backslash\{0\}$.

For the real vector bundle F, the corresponding metric is usually called the Euclidean metric.

Definition 1.2.5. The Euclidean metric is a smooth family $\{g_x^F\}_{x\in M}$ of bilinear maps $g_x^F : F_x \times F_x \to \mathbb{R}$ such that $g_x^F(\xi,\xi) > 0$ for any $\xi \in F_x\backslash\{0\}$.

Proposition 1.2.6. *There always exist Hermitian metrics on E.*

Proof. For any $U_i \times \mathbb{C}^m$, we could easily construct a smooth family of Hermitian products h_i^E on each fiber, e.g., taking the classical Hermitian product on $\mathbb{C}^m$. Let $\{\varphi_i\}$ be a partition of unity with respect to $\{U_i\}$. Then $\sum_i \varphi_i h_i^E$ is a Hermitian metric. $\square$

Similarly, there always exist Euclidean metric on real vector bundles.

Definition 1.2.7. A Euclidean metric on TM is called a Riemannian metric. A manifold with a Riemannian metric is called a Riemannian manifold.

Remark that the Hermitian (Euclidean) metric is far from unique.

Let h^E be a Hermitian metric on E. Now for $f, g \in \mathscr{C}^\infty(M, E)$, the Hermitian product is defined by

$$\langle f, g \rangle := \int_M h_x^E(f(x), g(x)) dv_x. \tag{1.2.15}$$

We denote the norm by

$$\|f\|_{L^2}^2 := \langle f, f \rangle. \tag{1.2.16}$$

Let $L^2(M, E)$ be the completion of the set $\{f \in \mathscr{C}^\infty(M, E) : \|f\|_{L^2} < +\infty\}$ with respect to the norm $\| \cdot \|_{L^2}$. It is also a Hilbert space. Similarly, we could denote the set of sections with compact support by $\mathscr{C}_0^\infty(M, E)$ and $\mathscr{C}_0^\infty(M, E)$ is dense in $L^2(M, E)$ with respect to the norm $\| \cdot \|_{L^2}$.

Once we extend the set of sections to a Hilbert space, naively, we want to extend the differential operator to

$$P : L^2(M, E) \to L^2(M, F) \tag{1.2.17}$$

If P is bounded, we could use a whole theory of functional analysis we learned to study the differential operator.

Unfortunately, the world is not as good as we think.

We will see this from the easiest differential operator: The derivative $\frac{d}{dt}$ on $\mathscr{C}_0^1(\mathbb{R})$.

Proposition 1.2.8. *The derivative $\frac{d}{dt}$ is unbounded with respect to the norm $\| \cdot \|_{L^2}$.*

Proof. Let $\varphi \in \mathscr{C}_0^\infty(\mathbb{R})$ such that $\|\varphi\|_{L^2} = 1$. Let $\|\frac{d\varphi}{dt}\|_{L^2} = C$. Then

$$\|n^{1/2}\varphi(nt)\|_{L^2}^2 = \int_{-\infty}^{+\infty} n\varphi^2(nt)dt = \int_{-\infty}^{+\infty} \varphi^2(t)dt = 1. \tag{1.2.18}$$

But

$$\begin{aligned} \left\| n^{1/2} \frac{d}{dt} \varphi(nt) \right\|_{L^2}^2 = \int_{-\infty}^{+\infty} n^3 \left(\frac{d\varphi}{dt}(nt) \right)^2 dt \\ = \int_{-\infty}^{+\infty} n^2 \left(\frac{d\varphi}{dt}(t) \right)^2 dt = n^2 C. \end{aligned} \tag{1.2.19}$$

Thus $\frac{d}{dt}$ is unbounded. □

Definition 1.2.9. Let W, W' be two Banach spaces. Let $P : W \to W'$ be a linear operator with domain $D(P)$. We say P is a closed operator if for $x_n \in D(P)$, $x_n \to x$, $Px_n \to y$, we have $x \in D(P)$ and $y = Px$.

Proposition 1.2.10. *The derivative $\frac{d}{dt}$ is a closed operator.*

Proof. The domain of $\frac{d}{dt}$ is $\mathscr{C}_0^1(\mathbb{R})$. We take $x_n(t) \in \mathscr{C}_0^1(\mathbb{R})$ and $x_n(t) \to x(t) \in L^2(\mathbb{R})$, $\frac{dx_n(t)}{dt} \to y(t) \in L^2(\mathbb{R})$, then we have $x_n(t) \to \int_{-\infty}^t y(s)ds$. Thus $x(t) = \int_{-\infty}^t y(s)ds$. So $x(t) \in \mathscr{C}_0^1(\mathbb{R})$ and $\frac{dx(t)}{dt} = y(t)$. □

Theorem 1.2.11 (Closed graph theorem)**.** *Let W, W' be two Banach spaces. Let $P : W \to W'$ be a closed operator with domain $D(P)$. If $D(P)$ is closed, then P is bounded.*

This is my most hate theorem. It prevents us to extend the domain of the derivative operator to the whole $L^2(\mathbb{R})$. (If we could extend, closed graph theorem implies $\frac{d}{dt}$ is bounded, which is a contradiction with Proposition 1.2.8).

For a large class of differential operator, we will meet the same obstruction coming from the functional analysis.

In the history of the differential operator, there are two ways to overcome this obstruction, any of them is not easy:

(1) reduce the Hilbert spaces to smaller Banach spaces, called the Sobolev spaces, such that the differential operator is bounded on these Hilbert spaces with respect to the new norms, which is the main purpose of this section;

(2) define the differential operator on a dense subset of the Hilbert space, which is the main idea of our next chapter.

1.2.2 Sobolev space

In 19th century, Gauss studied the electrostatic field and posed a famous problem, called the Dirichlet problem, that for a domain $\Omega \subset \mathbb{R}^2$, find a solution $u(x, y) \in \mathscr{C}^2(\Omega) \cap \mathscr{C}^0(\overline{\Omega})$ of

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = f, & f \in \mathscr{C}^0(\partial\Omega). \end{cases} \tag{1.2.20}$$

Later, Riemann discussed this problem and stated the Dirichlet Principle. For

$$\begin{aligned} &I(u) = \int_\Omega |\nabla u|^2 dxdy, \\ &u \in A = \{u \in \mathscr{C}^1(\Omega) : u_x, u_y \in L^2, u|_{\partial\Omega} = f\}, \end{aligned} \tag{1.2.21}$$

$I(u) \geq 0$, thus $\inf_A I(u)$ exists. Riemann said there exists $u_0 \in A$ such that $u_0 = \inf_A I(u)$. Then u_0 is the solution of (1.2.20).

In 1870, Weierstrass posed a counter example to explain that $\min_A I(u)$ may not exist. Sometimes, we cannot take $u_0 \in A$ such that $u_0 = \inf_A I(u)$. Later, people recognized that even $\min_A I(u)$ exists, it may not have the enough regularity.

In 1900, Hilbert confirmed the Dirichlet Principle for the smooth boundary. In his point of view, this is a very important problem, so that he posed three problems about this in his famous 23 problems.

Later, Sobolev stated a strategy to handle this problem. Firstly, we complete A into a complete space $\overline{A}$ with respect to some norm. In the complete space $\overline{A}$, obviously $\min_{\overline{A}} = \inf_{\overline{A}}$. Thus we could get a minimum element $u_0 \in \overline{A}$. Then we could use other method to study the regularity of it. This complete space is called the Sobolev space.

As usual, we first discuss the Sobolev space on $\mathbb{R}^n$. Since our main purpose is to study the differential operator on manifold without boundary, we will not discuss the boundary condition.

Let $\mathscr{S}$ be the set of $\mathbb{C}^m$-valued smooth function u on $\mathbb{R}^n$ such that for any n-tuple α and $k \in \mathbb{N}$, there exists $C_{\alpha,k} > 0$, such that

$$\left|(1+|x|)^k \frac{\partial^{|\alpha|} u}{\partial x^\alpha}(x)\right| \leq C_{\alpha,k}. \tag{1.2.22}$$

It is called the Schwartz space or space of rapidly decreasing functions.

For $u \in \mathscr{S}$, we define a norm $\|u\|_k$ on this space by

$$\|u\|_k^2 = \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} \left|\frac{\partial^{|\alpha|} u}{\partial x^\alpha}(x)\right|^2 dx. \tag{1.2.23}$$

In this part, we assume that all functions are $\mathbb{C}^m$-valued.

Definition 1.2.12. The completion of $\mathscr{S}$ relative to the norm $\|\cdot\|_k$ is the Sobolev space $\mathbf{H}^k$.

In order to do more things, we recall and summarize the knowledge of Fourier analysis on $\mathbb{R}^n$.

Let $u \in L^1(\mathbb{R}^n)$. The Fourier transform $\hat{u}$ of u is defined by

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} u(x) dx. \tag{1.2.24}$$

We denote by

$$D^\alpha = i^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial x^\alpha}. \tag{1.2.25}$$

Proposition 1.2.13. *For $u \in \mathscr{S}$, we have*

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \tag{1.2.26}$$

$$\widehat{x^\alpha u}(\xi) = (-1)^{|\alpha|} D_\xi^\alpha \hat{u}(\xi), \tag{1.2.27}$$

and the Plancherel's formula

$$(u, v)_{L^2} = (\hat{u}, \hat{v})_{L^2}. \tag{1.2.28}$$

Recall that the convolution product $*$ is defined by

$$u * v(x) = \int_{\mathbb{R}^n} u(x-y)v(y)dy, \tag{1.2.29}$$

for any $u, v \in \mathscr{S}$.

Proposition 1.2.14. *For $u, v \in \mathscr{S}$, we have*

$$\widehat{u \cdot v} = \hat{u} * \hat{v}, \quad \widehat{u * v} = \hat{u} \cdot \hat{v}. \tag{1.2.30}$$

Proposition 1.2.15. *The Fourier transform defines an isomorphism $:\mathscr{S} \to \mathscr{S}$. The inverse is given by*

$$u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \hat{u}(\xi) d\xi. \tag{1.2.31}$$

From (1.2.23), (1.2.26) and (1.2.28), we have

$$\begin{aligned} \|u\|_k^2 = \sum_{|\alpha|\le k} \|D^\alpha u\|_{L^2}^2 = \sum_{|\alpha|\le k} \|\widehat{D^\alpha u}\|_{L^2}^2 = \sum_{|\alpha|\le k} \|\xi^\alpha \hat{u}\|_{L^2}^2 \\ = \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} \xi^{2\alpha} \right) |\hat{u}(\xi)|^2 d\xi. \end{aligned} \tag{1.2.32}$$

Since there exist $c_1, c_2 > 0$ such that

$$c_1(1+|\xi|)^{2k} \le \sum_{|\alpha|\le k} \xi^{2\alpha} \le c_2(1+|\xi|)^{2k}, \tag{1.2.33}$$

$\|\cdot\|_k$ is equivalent to the norm $\|u\|_k'^2$ given by

$$\|u\|_k'^2 = \int_{\mathbb{R}^n} (1+|\xi|)^{2k} |\hat{u}(\xi)|^2 d\xi. \tag{1.2.34}$$

In some literatures, the weight part is $(1+|\xi|^2)^k$. The norm defined by this weight is also equivalent to $\|\cdot\|_k$.

In Functional analysis, the Hilbert spaces with the equivalent norms are topological isomorphism. Thus they could be regarded as the same space.

Following this way, we could define the Sobolev norm and the Sobolev space of any order s, $s \in \mathbb{R}$.

Definition 1.2.16. For $s \in \mathbb{R}$ and $u \in \mathscr{S}$, we define the s-th Sobolev norm $\|\cdot\|_s$ by

$$\|u\|_s^2 = \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi. \tag{1.2.35}$$

The completion of $\mathscr{S}$ with respect to this norm is the Sobolev space $\mathbf{H}^s$.

For $k \in \mathbb{N}$, the uniform $\mathscr{C}^k$-norm of $u \in \mathscr{C}^k$ is defined by

$$\|u\|_{\mathscr{C}^k}^2 := \sup_{\mathbb{R}^n} \sum_{|\alpha| \le k} |D^\alpha u|^2. \tag{1.2.36}$$

It is well-known that this norm is complete on any bounded domain.

Theorem 1.2.17 (Sobolev Embedding Theorem)**.** *For each $k \in \mathbb{N}$, $s > \frac{n}{2}+k$, $s \in \mathbb{R}$, there exists $C_s > 0$ such that for any $u \in \mathscr{S}$,*

$$\|u\|_{\mathscr{C}^k} \le C_s \|u\|_s. \tag{1.2.37}$$

Thus there exists a continuous embedding

$$\mathbf{H}^s \subset \mathscr{C}^k. \tag{1.2.38}$$

Proof. From (1.2.31) and (1.2.26), for $|\alpha| < s - \frac{n}{2}$,

$$\begin{aligned}
|D^\alpha u| &= (2\pi)^{-n/2} \left| \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{D^\alpha u}(\xi) d\xi \right| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |\xi^\alpha| |\hat{u}(\xi)| d\xi \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1+|\xi|)^{-(s-|\alpha|)} (1+|\xi|)^{(s-|\alpha|)} |\xi^\alpha| |\hat{u}(\xi)| d\xi \\
&\le (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{-2(s-|\alpha|)} d\xi \right)^{1/2} \left(\int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi \right)^{1/2}.
\end{aligned} \tag{1.2.39}$$

Since $s - |\alpha| > \frac{n}{2}$, $K_{\alpha,s} := \int_{\mathbb{R}^n} (1+|\xi|)^{-2(s-|\alpha|)} d\xi$ is finite. Thus we have

$$|D^\alpha u|^2 \le (2\pi)^{-n/2} K_{\alpha,s}^{1/2} \|u\|_s^2. \tag{1.2.40}$$

Let $C_s = (2\pi)^{-n/2} \sum_{|\alpha|<s-n/2} K_{\alpha,s}^{1/2}$, we get (1.2.37).

For any $u \in \mathbf{H}^s$, there exists a series $u_j \in \mathscr{S}$, such that $u_j \to u$ under the norm $\|\cdot\|_s$. Thus for any $\varepsilon > 0$, there exists $N > 0$ such that for any $j, l > N$, $\|u_j - u_l\|_s \leq \varepsilon/C_s$. By (1.2.37), we have $\|u_j - u_l\|_{\mathscr{C}^k} \leq \varepsilon$. Since $\mathscr{C}^k$ is complete on a bounded domain, there exists $u' \in \mathscr{C}^k$, such that $u_j \to u'$ under the norm $\|\cdot\|_{\mathscr{C}^k}$. We identify $u' \in \mathscr{C}^k$ with $u \in \mathbf{H}^s$ to get a continuous embedding $\mathbf{H}^s \subset \mathscr{C}^k$. □

From (1.2.35), if $s < t$, we have

$$\|u\|_s < \|u\|_t. \tag{1.2.41}$$

Thus we have the continuous embedding

$$\mathbf{H}^t \subset \mathbf{H}^s, \quad s < t. \tag{1.2.42}$$

Moreover, by Sobolev embedding theorem, for $0 < s_1 < \cdots < s_k < \cdots$,

$$\mathscr{C}_0^\infty \subset \mathscr{S} \subset \cdots \mathbf{H}^{s_k} \subset \cdots \mathbf{H}^{s_1} \subset L^2. \tag{1.2.43}$$

Theorem 1.2.18. *For $s_2 > s > s_1$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}^{s_2}$, we have*

$$\|u\|_s^2 \leq \varepsilon \|u\|_{s_2}^2 + C_\varepsilon \|u\|_{s_1}^2. \tag{1.2.44}$$

Proof. By (1.2.42), $\mathbf{H}^{s_2} \subset \mathbf{H}^s \subset \mathbf{H}^{s_1}$. Thus $u \in \mathbf{H}^s$ and $u \in \mathbf{H}^{s_1}$. From Definition 1.2.16, we only need to prove for any $\xi \in \mathbb{R}^n$, we have

$$(1 + |\xi|)^{2s} \leq \varepsilon (1 + |\xi|)^{2s_2} + C_\varepsilon (1 + |\xi|)^{2s_1}. \tag{1.2.45}$$

Let $\rho = (1 + |\xi|)^2 > 0$. We need to prove $\rho^s \leq \varepsilon \rho^{s_2} + C_\varepsilon \rho^{s_1}$. Set $\lambda = \varepsilon^{1/(s_2 - s)}$, $C_\varepsilon = \lambda^{-(s-s_1)}$. It is equivalent to $(\lambda\rho)^{s_2 - s} + (\lambda\rho)^{s - s_1} \geq 1$. This inequality holds because if $\lambda\rho \geq 1$, the first part ≥ 1, if $\lambda\rho < 1$, the second part < 1.

The proof of Theorem 1.2.18 is completed. □

From Theorems 1.2.17 and 1.2.18, we have the following corollary.

Corollary 1.2.19. *For $k \in \mathbb{N}$, $s > \frac{n}{2} + k$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}^s$, we have*

$$\|u\|_{\mathscr{C}^k} \leq \varepsilon \|u\|_s^2 + C_\varepsilon \|u\|_0^2. \tag{1.2.46}$$

Theorem 1.2.20 (Rellich Lemma)**.** *Let $\{u_j\}$ be a sequence of functions with $\operatorname{supp}(u_j) \subset B_1(0)$ and there exists constant $C > 0$, such that $\|u_j\|_t \leq C$. Then for any $s < t$, there exists a Cauchy subsequence of $\{u_j\}$ with respect to the norm $\|\cdot\|_s$, thus converges in $\mathbf{H}^s$.*

Proof. Take a smooth function $\varphi \in \mathscr{C}_0^\infty$ such that $\varphi = 1$ on B^n. Thus $\varphi u_j = u_j$. By (1.2.30), $\hat{u}_j = \hat{\varphi} * \hat{u}_j$. So for any α,

$$D_\xi^\alpha \hat{u}_j(\xi) = \int_{\mathbb{R}^n} (D_\xi^\alpha \hat{\varphi})(\xi - \eta)\hat{u}_j(\eta)d\eta. \tag{1.2.47}$$

By the Cauchy-Schwarz inequality,

$$|D_\xi^\alpha \hat{u}_j(\xi)|^2 \leq \int_{\mathbb{R}^n} (1 + |\eta|)^{-2t} |D_\xi^\alpha \hat{\varphi}|^2(\xi - \eta)d\eta \times \int_{\mathbb{R}^n} (1 + |\eta|)^{2t} |\hat{u}_j(\eta)|^2 d\eta = C_\alpha(\xi)\|u_j\|_t^2, \tag{1.2.48}$$

where $C_\alpha(\xi) = \int_{\mathbb{R}^n}(1 + |\eta|)^{-2t}|D_\xi^\alpha\hat{\varphi}|^2(\xi - \eta)d\eta$ is finite, since $\hat{\varphi} \in \mathscr{S}$, and does not depend on u.

By (1.2.48), for any α, $|D_\xi^\alpha \hat{u}_j(\xi)|$ is uniformly bounded on compact subset of $\mathbb{R}^n$. Thus $\{\hat{u}_j\}$ is uniformly equicontinuous on compact subsets. By Ascoli-Arzela Theorem, there is a subsequence of $\{\hat{u}_j\}$ which is uniformly Cauchy on compact subsets, which we also denote by $\{\hat{u}_j\}$.

Fix $r > 0$.

$$\|u_j - u_k\|_s^2 = \int_{|\xi|>r} (1 + |\xi|)^{2s} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 d\xi + \int_{|\xi|\leq r} (1 + |\xi|)^{2s} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 d\xi =: A + B. \tag{1.2.49}$$

Note that

$$A \leq (1 + r)^{2(s-t)} \int_{|\xi|>r} (1 + |\xi|)^{2t} |\hat{u}_j(\xi) - \hat{u}_j(\xi)|^2 d\xi \leq \frac{\|u_j - u_k\|_t^2}{(1 + r)^{2(t-s)}} \leq \frac{2C}{(1 + r)^{2(t-s)}}. \tag{1.2.50}$$

For any $\varepsilon > 0$, we take r large, such that $2C(1 + r)^{2(s-t)} < \varepsilon/2$. Note that

$$B \leq C' \sup_{|\xi|<r} |\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2. \tag{1.2.51}$$

Since $\{\hat{u_j}\}$ is uniformly Cauchy, there exists $N > 0$ such that for $j, k > N$, $|\hat{u}_j(\xi) - \hat{u}_k(\xi)|^2 \leq \varepsilon/(2C')$. Thus for $j, k > N$, $\|u_j - u_k\|_s^2 < \varepsilon$.

The proof of Theorem 1.2.20 is completed. □

By Theorems 1.2.17, 1.2.20, we have the following corollary.

Corollary 1.2.21. *Let* $\{u_j\}$ *be a sequence of functions with* $\mathrm{supp}(u_j) \subset B_1(0)$ *and there exists constant* $C > 0$*, such that* $\|u_j\|_s \leq C$*. If* $s > \frac{n}{2} + k$*, then there is a subsequence which converges to a function* $u \in \mathscr{C}_0^k$ *in the uniform* $\mathscr{C}^k$*-norm.*

The following theorem says that $\mathbf{H}^{-s} = (\mathbf{H}^s)^*$.

Theorem 1.2.22. *For* $u, v \in \mathscr{S}$*, the bilinear function*

$$(u, v) := \int_{\mathbb{R}^n} \hat{u}(\xi) \cdot \hat{v}(\xi) d\xi \tag{1.2.52}$$

has a continuous extension to $\mathbf{H}^s \times \mathbf{H}^{-s}$ *for any* $s \in \mathbb{R}$*. Moreover, it identifies* $\mathbf{H}^{-s}$ *with the dual of* $\mathbf{H}^s$*, that is,*

$$\|v\|_{-s} = \sup_{u \in \mathbf{H}^s} \frac{(u, v)}{\|u\|_s}. \tag{1.2.53}$$

Proof. For $u, v \in \mathscr{S}$, by the Schwarz inequality,

$$|(u, v)| = \left| \int_{\mathbb{R}^n} (1 + |\xi|)^s \hat{u}(\xi) \cdot (1 + |\xi|)^{-s} \hat{v}(\xi) d\xi \right| \leq \|u\|_s \|v\|_{-s}. \tag{1.2.54}$$

From the argument below (1.2.40), we see that the bilinear function $(\cdot, \cdot)$ has a continuous extension to $\mathbf{H}^s \times \mathbf{H}^{-s}$ for any $s \in \mathbb{R}$.

We choose u such that $\hat{u}(\xi) = \overline{\hat{v}(\xi)}(1 + |\xi|)^{-2s}$. Then

$$\|u\|_s = \|v\|_{-s}, \quad (u, v) = \int_{\mathbb{R}^n} |\hat{v}(\xi)|^2 (1 + |\xi|)^{-2s} d\xi = \|v\|_{-s}^2. \tag{1.2.55}$$

Thus

$$\sup_{u \in \mathbf{H}^s} \frac{(u, v)}{\|u\|_s} \geq \|v\|_{-s}. \tag{1.2.56}$$

Then (1.2.53) follows from (1.2.54) and (1.2.56).

The proof of Theorem 1.2.22 is completed. □

Corollary 1.2.23. *Let* $T : \mathscr{S} \to \mathscr{S}$ *and* $T^* : \mathscr{S} \to \mathscr{S}$ *be linear maps such that* $(Tu, v) = (u, T^*v)$ *for any* $u, v \in \mathscr{S}$*. For* $s \in \mathbb{R}$*, if there exists* $C > 0$ *such that for any* $u \in \mathscr{S}$*,*

$$\|Tu\|_s \leq C\|u\|_s, \tag{1.2.57}$$

we have

$$\|T^*u\|_{-s} \leq C\|u\|_{-s}. \tag{1.2.58}$$

Proof. For $u, v \in \mathscr{S}$, by Theorem 1.2.22, we have

$$\|T^*v\|_{-s} = \sup_{u\in\mathbf{H}^s} \frac{(u, T^*v)}{\|u\|_s} = \sup_{u\in\mathbf{H}^s} \frac{(Tu, v)}{\|u\|_s} \\ \le \sup_{u\in\mathbf{H}^s} \frac{\|Tu\|_s\|v\|_{-s}}{\|u\|_s} \le C\|v\|_{-s}. \quad (1.2.59)$$

The proof of Corollary 1.2.23 is completed. □

Proposition 1.2.24. *Let A be a smooth matrix valued function on $\mathbb{R}^n$ such that $|D^\alpha A|$ is bounded for any α. Then the map $T : \mathscr{S} \to \mathscr{S}$ defined by $Tu = Au$ extends to a bounded linear map $T : \mathbf{H}^s \to \mathbf{H}^s$ for $s \in \mathbb{Z}$.*

Proof. For $s \ge 0$, there exists $C > 0$, such that for any $u \in \mathscr{S}$,

$$\|Tu\|_s = \sum_{|\alpha|\le s} \int_{\mathbb{R}^n} |D^\alpha(Au)|^2 dx \le C \sum_{|\alpha|\le s} \int_{\mathbb{R}^n} |D^\alpha u|^2 dx = C\|u\|_s. \quad (1.2.60)$$

By (1.2.28), we see that $T^*u = A^T u$. For $s < 0$, as in (1.2.60), we have $\|T^*u\|_{-s} \le C\|u\|_{-s}$. Since $(T^*)^* = T$, by Corollary 1.2.23, we have $\|Tu\|_s \le C\|u\|_s$.

The proof of Proposition 1.2.24 is completed. □

Now we want to establish Proposition 1.2.24 for $s \in \mathbb{R}$. We prove a lemma first.

Lemma 1.2.25 (Peetre's Inequality)**.** *For any $\xi, \eta \in \mathbb{R}^n$ and $s \in \mathbb{R}$, we have*

$$\left(\frac{1+|\xi|}{1+|\eta|}\right)^s \le (1+|\xi-\eta|)^{|s|}. \quad (1.2.61)$$

Proof. For $s \ge 0$, (1.2.61) follows from

$$(1+|\xi|) \le 1+|\xi-\eta|+|\eta| \le (1+|\eta|)(1+|\xi-\eta|). \quad (1.2.62)$$

For $s < 0$, we use the same argument reversing ξ and η and replace s by $-s$. □

For a smooth matrix valued function $A(x) = [a_{ij}(x)]$ on $\mathbb{R}^n$, we say $A \in \mathscr{S}$ if for any i, j, $a_{ij}(x) \in \mathscr{S}$. Then we could define $\widehat{A}(\xi) = [\widehat{a_{ij}}(\xi)]$. In this case, for $u \in \mathscr{S}$, letting $A * u = \int_{\mathbb{R}^n} A(x-y)u(y)dy$, by Proposition 1.2.14, we have

$$\widehat{Au} = \widehat{A} * \hat{u}, \quad \widehat{A * u} = \widehat{A}\,\hat{u}. \quad (1.2.63)$$

Theorem 1.2.26. *Let $A \in \mathscr{S}$ be a smooth matrix valued function. Then the map $T : \mathscr{S} \to \mathscr{S}$ defined by $Tu = Au$ extends to a bounded linear map $T : \mathbf{H}^s \to \mathbf{H}^s$ for $s \in \mathbb{R}$, i.e., there exists $C > 0$ such that for any $u \in \mathscr{S}$, we have*

$$\|Au\|_s \leq C\|u\|_s. \tag{1.2.64}$$

Proof. For any $u \in \mathscr{S}$, by (1.2.35) and (1.2.63), we have

$$\begin{aligned}\|Au\|_s^2 &= \int_{\mathbb{R}^n} (1+|\xi|)^{2s} |\widehat{Au}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1+|\xi|)^{2s} \left| \int_{\mathbb{R}^n} \widehat{A}(\xi-\eta)\hat{u}(\eta)d\eta \right|^2 d\xi \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{(1+|\xi|)^{2s}}{(1+|\eta|)^{2s}} |\widehat{A}(\xi-\eta)|^2 d\xi \right) |(1+|\eta|)^{2s} \hat{u}(\eta)|^2 d\eta. \end{aligned} \tag{1.2.65}$$

From (1.2.22) and Lemma 1.2.25, for $k > |s| + \frac{n}{2}$, $k \in \mathbb{N}$, there exists $C_k > 0$, such that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{(1+|\xi|)^{2s}}{(1+|\eta|)^{2s}} |\widehat{A}(\xi-\eta)|^2 d\xi &\leq C_k \int_{\mathbb{R}^n} (1+|\xi-\eta|)^{2|s|-2k} d\xi \\ &= C_k \int_{\mathbb{R}^n} (1+|\xi|)^{2|s|-2k} d\xi < +\infty. \end{aligned} \tag{1.2.66}$$

Let $C = C_k \int_{\mathbb{R}^n} (1+|\xi|)^{2|s|-2k} d\xi$. By (1.2.65),

$$\|Au\|_s \leq C\|u\|_s. \tag{1.2.67}$$

The proof of Theorem 1.2.26 is completed. □

In order to define the Sobolev space on manifolds, we introduce another equivalent Sobolev norm for $s \in \mathbb{R}$, which is due to Hörmander.

Proposition 1.2.27. *For $0 < s < 1$, the s-th Sobolev norm is equivalent to the following norm for any $u \in \mathscr{S}$:*

$$\|u\|_s' := \left(\|u\|_{L^2}^2 + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}. \tag{1.2.68}$$

Proof. By Newton-Leibniz's formula,

$$u(x) - u(y) = (x-y) \cdot \int_0^1 \nabla u(tx + (1-t)y)dt. \tag{1.2.69}$$

By (1.2.69), since $s < 1$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy\\
&\qquad\le C\sum_{|\alpha|=1}\cdot\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\int_0^1|(\nabla u)(tx+(1-t)y)|dt\right)^2}{|x-y|^{n+2s-2}}dxdy\\
&\qquad= C\sum_{|\alpha|=1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\left(\int_0^1|(\nabla u)(tx+y)|dt\right)^2}{|x|^{n+2s-2}}dxdy\\
&\le C\sum_{|\alpha|=1}\int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s-2}}\int_0^1\int_{\mathbb{R}^n}|(\nabla u)(tx+y)|^2dydtdx<+\infty. \qquad (1.2.70)
\end{aligned}$$

From (1.2.24), letting $u_x(y) := u(x+y)$, we have

$$\widehat{u_x}(\xi) = e^{i\langle x,\xi\rangle}\hat{u}(\xi). \qquad (1.2.71)$$

Thus from Plancherel's formula (1.2.28) and (1.2.71), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy=\int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s}}\int_{\mathbb{R}^n}|u(x+y)-u(y)|^2dydx\\
&=\int_{\mathbb{R}^n}\frac{1}{|x|^{n+2s}}\int_{\mathbb{R}^n}|\widehat{u_x}(\xi)-\hat{u}(\xi)|^2dydx=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|e^{i\langle x,\xi\rangle}-1|^2}{|x|^{n+2s}}|\hat{u}(\xi)|^2dxd\xi.
\end{aligned} \qquad (1.2.72)$$

By replacing ξ to $T\xi$, where T is an orthogonal rotation, we see that $\int_{\mathbb{R}^n}|e^{i\langle x,\xi\rangle}-1|^2|x|^{-n-2s}dx$ depends only on $|\xi|$. By replacing ξ to $a\xi$, $a\in\mathbb{R}$, we see that it is homogeneous of degree $2s$. Thus by (1.2.93),

$$\int_{\mathbb{R}^n}|e^{i\langle x,\xi\rangle}-1|^2|x|^{-n-2s}dx = C_s|\xi|^{2s}, \quad C_s>0. \qquad (1.2.73)$$

Therefore, we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy = C_s\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi. \qquad (1.2.74)$$

Since there exist $c, C>0$ such that $c(1+|\xi|^{2s})\le(1+|\xi|)^{2s}\le C(1+|\xi|^{2s})$, we see the two norms are equivalent.

The proof of Proposition 1.2.27 is completed. □

Corollary 1.2.28. *For $s > 0$, $s = m + \sigma$ with $m \in \mathbb{N}$, $0 \le \sigma < 1$, the s-th Sobolev norm is equivalent to the following norm for any $u \in \mathscr{S}$:*

$$\|u\|'_s := \left(\sum_{|\alpha| \le m} \|D^\alpha u\|^2_{L^2} + \sum_{|\alpha| \le m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dx dy \right)^{\frac{1}{2}}. \tag{1.2.75}$$

Proof. By (1.2.74), we have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} dx dy &= C_\sigma \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\widehat{D^\alpha u}(\xi)|^2 d\xi \\ &= C_\sigma \int_{\mathbb{R}^n} |\xi|^{2(\alpha+\sigma)} |\widehat{u}(\xi)|^2 d\xi. \end{aligned} \tag{1.2.76}$$

Then Corollary 1.2.28 follows directly. □

For $s < 0$, as in (1.2.53), we define

$$\|u\|'_s = \sup_{v \in \mathbf{H}^{-s}} \frac{(u, v)}{\|v\|'_{-s}}. \tag{1.2.77}$$

Proposition 1.2.29. *Let Ω and Ω' be bounded open subsets of $\mathbb{R}^n$. Let $\Phi : \Omega \to \Omega'$ be a diffeomorphism. Let $K \subset \Omega$ be a compact subset and $K' = \Phi(K)$. Then for $s \in \mathbb{R}$, there exists $c, C > 0$ such that for any $u \in \mathscr{C}_0^\infty(K')$,*

$$c\|u\|_s \le \|u \circ \Phi\|_s \le C\|u\|_s. \tag{1.2.78}$$

Proof. Since Φ is a diffeomorphism, we only need to prove

$$\|u \circ \Phi\|_s \le C\|u\|_s \tag{1.2.79}$$

because the other inequality follows from considering Φ^{-1}.

We denote by $U = u \circ \Phi$. Let $|D\Phi^{-1}|$ be the Jacobian determinant of Φ^{-1}. Set

$$B_1 = \sup_{K_2} |D\Phi^{-1}|, \quad B_2 = \sup_{K_1} \frac{|\Phi(x) - \Phi(y)|}{|x-y|}. \tag{1.2.80}$$

Set $x' = \Phi(x)$, $y' = \Phi(y)$. Then for $s = 0$, (1.2.79) follows from

$$\int_{\mathbb{R}^n} |U(x)|^2 dx = \int_{\mathbb{R}^n} |u(x')|^2 |D\Phi^{-1}| dx' \le B_1 \int_{\mathbb{R}^n} |u(x)|^2 dx. \tag{1.2.81}$$

For $0 < s < 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x')-u(y')|^2}{|x-y|^{n+2s}}|D\Phi^{-1}(x')||D\Phi^{-1}(y')|dx'dy' \\ &\le B_1^2 B_2^{n+2s}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x')-u(y')|^2}{|x'-y'|^{n+2s}}dx'dy'. \quad (1.2.82)\end{aligned}$$

Then by Proposition 1.2.27, we have $\|u\circ\Phi\|_s \le C\|u\|_s$.

Now we proceed by induction. Let $\chi' \in \mathscr{C}_0^\infty(\Omega')$ such that $\chi' \equiv 1$ on K'. Assume that (1.2.79) holds for any $0 \le s < k$, $k \in \mathbb{N}$. For $k \le s < k+1$, $1 \le j \le n$, by Theorem 1.2.26, we have

$$\begin{aligned}\|D^j U(x)\|_{s-1} \le \|D^j u(x')D^j\Phi(x)\|_{s-1} &= \|D^j u(x')\chi'(x')D^j\Phi(x)\|_{s-1} \\ &\le C\|D^j u(x')\|_{s-1}. \quad (1.2.83)\end{aligned}$$

Since $1+|\xi|^2 \le (1+|\xi|)^2 \le 2(1+|\xi|^2)$, we have

$$\|u\|_{s-1}^2 + \sum_{j=1}^n \|D^j u\|_{s-1}^2 \le \|u\|_s^2 \le 2\|u\|_{s-1}^2 + 2\sum_{j=1}^n \|D^j u\|_{s-1}^2. \quad (1.2.84)$$

From (1.2.83), (1.2.84) and the assumption for the induction, we have

$$\begin{aligned}\|U\|_s^2 &\le 2\|U\|_{s-1}^2 + 2\sum_{j=1}^n \|D^j U\|_{s-1}^2 \\ &\le C\|u\|_{s-1}^2 + C\sum_{j=1}^n \|D^j u\|_{s-1}^2 \le C\|u\|_s^2. \quad (1.2.85)\end{aligned}$$

Therefore, (1.2.79) holds for any $s \ge 0$.

For $s < 0$, we use the duality of $\mathbf{H}^s$ and $\mathbf{H}^{-s}$. Let $\chi \in \mathscr{C}_0^\infty(\Omega')$ such that $\chi \equiv 1$ on K. Then by (1.2.56) and Theorem 1.2.26, for any $v \in \mathscr{S}$, we have

$$\begin{aligned}|(v,U)| = |(\chi_1 v, U)| = |(\chi v\circ\Phi^{-1}, u|D\Phi|)| &\le \|\chi v\circ\Phi^{-1}\|_{-s}\||D\Phi|u\|_s \\ &\le \|\chi v\circ\Phi^{-1}\|_{-s}\|\chi'|D\Phi|u\|_s \le C\|v\|_{-s}\|u\|_s. \quad (1.2.86)\end{aligned}$$

Therefore,

$$\|u\circ\Phi\|_s = \sup_{v\in\mathscr{S}}\frac{|(v,U)|}{\|v\|_{-s}} \le C\|u\|_s. \quad (1.2.87)$$

The proof of Proposition 1.2.29 is completed.

□

Now we start to define the Sobolev space on the manifold.

Definition 1.2.30. Let M be a Riemannian manifold and E be a vector bundle over M with Hermitian metric. Let $K \subset M$ be a compact subset. Let $\{U_\alpha, \phi_\alpha\}$ be a locally finite atlas of M such that E is trivial on U_i and K is covered by finite charts. Let $\{h_\alpha\}$ be a partition of unity. Then for $s \in \mathbb{R}$, the Sobolev s-norm can be defined on $u \in \mathscr{C}_0^\infty(K, E)$ by

$$\|u\|_s^2 := \sum_\alpha \|(h_\alpha u) \circ \phi_\alpha^{-1}\|_s^2. \tag{1.2.88}$$

The completion of $\mathscr{C}_0^\infty(M, E)$ in this norm is the Sobolev $\mathbf{H}_0^s(K, E)$. If M is compact, we set $\mathbf{H}^s(M, E) := \mathbf{H}_0^s(M, E)$. We often denote it by $\mathbf{H}^s(E)$ if there is no confusion.

This norm is of course highly non-intrinsic. However, Theorem 1.2.26 shows that $\|\cdot\|_s$ is independent of the choice of local chart and partition of unity up to equivalence. Proposition 1.2.29 shows that $\|\cdot\|_s$ is independent of the choice of coordinate transformation up to equivalence.

Proposition 1.2.31. *The equivalence class of the norm $\|\cdot\|_s$ is independent of the atlas and the partition of unity.*

Proof. Let $\{V_\lambda, \psi_\lambda\}$ be a locally finite atlas of M such that E is trivial on V_λ and K is covered by finite charts.. Let $\{g_\lambda\}$ be a partition of unity with respect to this new atlas. Note that if $U_\alpha \cap V_\lambda \neq \emptyset$, we see that $\operatorname{supp}(h_\alpha \cdot g_\lambda) \subset U_\alpha \cap V_\lambda$. By Theorem 1.2.26 and Proposition 1.2.29, for $u \in \mathscr{C}^\infty(K, E)$, we have

$$\begin{aligned}\sum_\lambda \|(g_\lambda \cdot u) \circ \psi_\lambda^{-1}\|_s^2 &= \sum_\lambda \left\| \left(\sum_\alpha h_\alpha \cdot g_\lambda \cdot u \right) \circ \psi_\lambda^{-1} \right\|_s^2 \\ \leq C \sum_{\lambda,\alpha} \left\|(h_\alpha \cdot g_\lambda \cdot u) \circ \psi_\lambda^{-1}\right\|_s^2 &= C \sum_{\lambda,\alpha} \left\|(h_\alpha \cdot g_\lambda \cdot u) \circ \phi_\alpha^{-1} \circ (\phi_\alpha \circ \psi_\lambda^{-1})\right\|_s^2 \\ \leq C \sum_{\lambda,\alpha} \left\|(g_\lambda \cdot h_\alpha \cdot u) \circ \phi_\alpha^{-1}\right\|_s^2 &\leq C \sum_\alpha \left\|(h_\alpha \cdot u) \circ \phi_\alpha^{-1}\right\|_s^2.\end{aligned} \tag{1.2.89}$$

The proof of Proposition 1.2.31 is completed. □

For the setting in Definition 1.2.30, we define the $\mathscr{C}^k$-norm by

$$\|u\|_{\mathscr{C}^k}^2 := \sum_\alpha \|(h_\alpha u) \circ \phi_\alpha^{-1}\|_{\mathscr{C}^k}^2. \tag{1.2.90}$$

As in Proposition 1.2.31, the equivalence class of the norm $\|\cdot\|_{\mathscr{C}^k}$ is independent of the atlas and the partition of unity.

Let ∇ be a connection on E. Given $u \in \mathscr{C}^\infty(M, E)$, we have $\nabla u \in \mathscr{C}^\infty(M, T^*M \otimes E)$. Using the tensor product connection on $T^*M \otimes E$, we have $\nabla\nabla u \in \mathscr{C}^\infty(M, T^*M \otimes T^*M \otimes E)$. This process continues, for any $s \in \mathbb{N}$, we define

$$\|u\|_k''^2 := \sum_{j=1}^k \int_M |\underbrace{\nabla \cdots \nabla}_{j \text{ times}} u|^2 dv_x. \tag{1.2.91}$$

Proposition 1.2.32. *The norm $\|\cdot\|_k''$ in (1.2.91) is equivalent to $\|\cdot\|_k$ in (1.2.60) for $k \in \mathbb{N}$.*

Proof. By (1.2.91), we have

$$\begin{aligned}\|u\|_k''^2 := \sum_{j=1}^k \sum_\lambda \int_{\phi_\lambda(V_\lambda)} & g_\lambda \circ \psi_\lambda^{-1} \\ & \cdot \left| \sum_{1 \le i_1, \cdots, i_j \le n} (D^{i_1} + A_{i_1}) \cdots (D^{i_j} + A_{i_j})(u \circ \psi_\lambda^{-1}) \right|^2 dv_x. \end{aligned} \tag{1.2.92}$$

Then Proposition 1.2.32 follows from (1.2.23) and (1.2.88). □

From Proposition 1.2.32, the norm in (1.2.91) is independent of the metrics on TM and E and the connection ∇ up to equivalence. Sometimes, we use (1.2.91) as the definition of the Sobolev norm.

For $k \in \mathbb{N}$, the uniform $\mathscr{C}^k$-norm of $u \in \mathscr{C}_0^k(K, E)$ is defined by

$$\|u\|_{\mathscr{C}^k}''^2 := \sup_K \sum_{|\alpha| \le k} |\underbrace{\nabla \cdots \nabla}_{j \text{ times}} u|^2. \tag{1.2.93}$$

As in Proposition 1.2.32, this norm is equivalent to that in (1.2.90). Since K is compact, this norm is complete.

Now we generalize Theorems 1.2.17, 1.2.18, 1.2.20, 1.2.22 and Proposition 1.2.24 to the global case. The proof of the following theorem is obvious, which is left as an exercise.

Theorem 1.2.33. *Let E and F be vector bundles over a manifold of dimension n.*

(1) (Rellich's theorem) For any $s, t \in \mathbb{R}$, $s < t$, the inclusion map

$$\iota : \mathbf{H}_0^t(K, E) \to \mathbf{H}_0^s(K, E) \tag{1.2.94}$$

is a compact operator, i.e., ι sends any bounded subset of $\mathbf{H}_0^t(K,E)$ to relatively compact subset of $\mathbf{H}_0^s(K,E)$, equivalently, a set with compact closure.

(2) (Sobolev embedding theorem) For each $k \in \mathbb{N}$ and each $s > \frac{n}{2} + k$, there is a continuous inclusion $\mathbf{H}_0^s(K,E) \subset \mathscr{C}_0^k(K,E)$, that is, for any $u \in \mathscr{C}_0^\infty(K,E)$,

$$\|u\|_{\mathscr{C}^k} \leq C_s \|u\|_s. \tag{1.2.95}$$

Furthermore, by (1), every sequence $\{u_j \in \mathscr{C}_0^\infty(K,E)\}$ which is bounded in the $\|\cdot\|_s$ norm has a subsequence which converges in the uniform $\mathscr{C}^k$-norm.

(3) For $s_2 > s > s_1$, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$, such that for any $u \in \mathbf{H}_0^{s_2}(K,E)$, we have

$$\|u\|_s^2 \leq \varepsilon \|u\|_{s_2}^2 + C_\varepsilon \|u\|_{s_1}^2. \tag{1.2.96}$$

In particular, for $s > \frac{n}{2} + k$, $k \geq 1$,

$$\|u\|_{\mathscr{C}^k}^2 \leq \varepsilon \|u\|_s^2 + C_\varepsilon \|u\|_0^2. \tag{1.2.97}$$

(4) For any Riemannian volume element dv on compact manifold M, the bilinear map on $\mathscr{C}^\infty(M,E) \times \mathscr{C}^\infty(M,E^*)$ given by setting

$$(u, v^*) = \int_M v^*(u) dv \tag{1.2.98}$$

has a continuous extension to $\mathbf{H}^s(E) \times \mathbf{H}^{-s}(E^*)$ for any $s \in \mathbb{R}$. Moreover, it identifies $\mathbf{H}^{-s}(E^*)$ with the dual of $\mathbf{H}^s(E)$, that is,

$$\|v^*\|_{-s} = \sup_{u \in \mathbf{H}^s(E)} \frac{(u, v^*)}{\|u\|_s}. \tag{1.2.99}$$

(5) Multiplication $T_A u := Au$ by any $A \in \mathscr{C}_0^\infty(K, \mathrm{Hom}(E,F) := F \otimes E^*)$ extends to a bounded linear map $T_A : \mathbf{H}_0^s(K,E) \to \mathbf{H}_0^s(K,F)$ for all $s \in \mathbb{R}$.